(09.02.21)

CLASS: 13

CONVERG. OF FOURIER SERIES

*) THURSDAY: 10^{15} - 12^{00} in GK1 see website

*) THURSDAY: 6.15 — 7.00

11/02 $\begin{cases} \text{4 - 5 in office 9\_\_} \\ \text{5 - 5.45 ZOOM} \end{cases}$ Remember mask!

Q: How often new exercises?

Dep. on when we finish next chapter

Next week on sunday ch 3 exercises

Then Ch 5: two lists?

Theorem: f is integrable on the circle, THEN

$$\lim_{N\to\infty} \underbrace{\|S_N(f) - f\|}_{\frac{1}{2\pi}\int_0^{2\pi}|S_N(f)(x) - f(x)|^2 dx \to 0} = 0$$

Approximation lemma: f is integrable on the circle:

Then: $\|f - S_N(f)\| \le \|f - \sum_{|k|\le N} b_k e_k\|$

$b_k \in \mathbb{C}$.

Proof: [try to prove for cont. funcs first. Then use approx lemma.]

i) Assume that f continuous. ($f:[0,2\pi]\to\mathbb{C}$.)

*) $|f(x)|\leq B$, then, using approximation lemma, $\forall\varepsilon>0$

$\exists P$: Trig polynomial s.t.

$$|f(x)-P(x)|^2<\varepsilon^2 \quad \forall x\in[0,2\pi].$$

$$\underbrace{\frac{1}{2\pi}\int_0^{2\pi}|f(x)-P(x)|^2dx}_{\|f-P\|^2}<\varepsilon^2$$

$$\|f-P\|^2<\varepsilon^2 \longrightarrow P \text{ has degree } "M=M(\varepsilon)".$$

Let $N\geq M \Rightarrow \|f-S_N(f)\|\leq\|f-P\|<\varepsilon^2$

$$\lim_{N\to\infty}\|f-S_N(f)\|=0.$$

ii) f integrable, $|f(x)|\leq B$. Using approximation lemma 1 [approx integrable func w/ cont funcs.]

$\exists\{f_k\}$ cont. $|f_k(x)|\leq B$

$$\lim_{k\to\infty}\int_0^{2\pi}|f(x)-f_k(x)|dx=0.$$

[2: approx cont. w/ trig polynomial.]

[3: fourier series is best trig polynom approx.]

In analysis: classic technique prove for nice space, then extend.

Let $\varepsilon>0$ $\exists k_0\in\mathbb{N}$: $\Rightarrow \int_0^{2\pi}|f(x)-f_{k_0}(x)|dx<\varepsilon$.

$\forall \varepsilon > 0\ \exists \{f_k\}$ cont. $\quad |f_k(x)| \le B.$

$$\int_0^{2\pi} |f(x) - f_{k_0}(x)|\, dx < \varepsilon$$

$$\|f - f_{k_0}\|^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(x) - f_{k_0}(x)|^2 dx = \frac{1}{2\pi}\int_0^{2\pi} |f(x) - f_{k_0}(x)| \cdot \underbrace{|f(x) - f_{k_0}(x)|}_{\le |f(x)| + |f_{k_0}(x)| \le 2B} dx$$

$$\le \frac{2B}{2\pi}\int_0^{2\pi} |f(x) - f_{k_0}(x)|\, dx$$

$$\le \frac{B}{\pi}\cdot \varepsilon^2 \Rightarrow \|f - f_{k_0}\| \le \sqrt{\frac{B}{\pi}}\,\varepsilon$$

For the function f_{k_0} (continuous):

$\exists P$: trig. polynomials of degree M,

$$\|f_{k_0} - P\| < \varepsilon$$

Using Δ inequality:

$$\|f - P\| \le \|f - f_{k_0}\| + \|f_{k_0} - P\|$$

$$< C\varepsilon + \varepsilon = D\varepsilon$$

Then, using approx lemma 3, for $N \geq M$:

$$\Rightarrow \|f - S_N(f)\| \le \|f - P\| \le D\varepsilon$$

and we are done!

BREAK

*) $\|f - S_N(f)\| \longrightarrow 0$

$$\|f\|^2 = \|f - S_N(f)\|^2 + \sum_{|n| \le N} |\hat{f}(n)|^2$$ ← Mentioned before.

[$\|f - \sum b_k e_k\| = \|f - S_N(f) + S_N(f) - \sum b_k e_k\|$ → Give you approx lemma 3.]

$\le \|f - S_N\| + \|S_N - \sum b_k e_k\|$

if: $N \to \infty$:

*) $$\boxed{\|f\|^2 = \sum_{|n| \le N} |\hat{f}(n)|^2 = \sum_{n \in \mathbb{Z}} |\hat{f}(n)|^2}$$

PARSEVAL'S IDENTITY.

$$\|f\|^2 = \frac{1}{2\pi} \int_0^{2\pi} |f(x)|^2 dx = \sum_{n \in \mathbb{Z}} |\hat{f}(n)|^2.$$

$\{e_n\}_{n \in \mathbb{Z}}$ $e_n(\theta) = e^{in\theta}$ → orthonormal family.

*) $\hat{f}(n) = \langle f, e_n \rangle$

*) $\langle f - S_N(f), e_n \rangle = 0$, $|n| \le N$

$$\begin{aligned} \langle f - \sum_m \hat{f}(m) e_m, e_n \rangle &= \langle f, e_n \rangle - \langle \sum_m \hat{f}(m) e_m, e_n \rangle \\ &= \langle f, e_n \rangle - \hat{f}(n) \underbrace{\langle e_n, e_n \rangle}_{=1} \\ &= \hat{f}(n) - \hat{f}(n) = 0 \end{aligned}$$

$*)\ \langle f - S_N(f), \sum_{|n|\le N} b_n e_n \rangle = 0$

$*)\ \langle f - S_N(f), S_N(f) \rangle = 0$

$*)\ \|f\|^2 = \|f - S_N(f)\|^2 + \|S_N(f)\|^2$ [Pythagoras]

$$\|S_N(f)\|^2 = \Big\| \sum_{|n|\le N} \hat{f}(n) e_n \Big\|^2 = \Big\| \sum_{|n|\le N} \langle f, e_n \rangle e_n \Big\|^2$$

$$= \sum_{|n|\le N} |\hat{f}(n)|^2$$

But can use any orthonormal family.

Replace: $e_n \to v_n$

$S_N(f) \to V_N(f)$

$\{V_n\}_{n\in\mathbb{Z}} \longrightarrow$ orthonormal family.

$e_n \to V_n$

$S_N(f) \to V_N(f)$

$*)\ a_n = \langle f, V_n \rangle$

$*)\ \hat{f}(n) = \langle f, e_n \rangle$

?

$*)\ \langle f - V_N(f), e_n \rangle = 0, \quad |n| \le N$

$$\Big[\langle f - \sum_m \hat{f}(m) e_m, e_n \rangle = \langle f, e_n \rangle - \langle \sum_m \hat{f}(m) e_m, e_n \rangle$$
$$= \langle f, e_n \rangle - \hat{f}(n) \underbrace{\langle e_n, e_n \rangle}_{=1}$$
$$= \hat{f}(n) - \hat{f}(n) = 0 \Big]$$

*) $\langle f - V_N(f), \sum_{|n|\leq N} b_n V_n \rangle = 0$

*) $\langle f - V_N(f), V_N(f) \rangle = 0$

*) $\|f\|^2 = \|f - V_N(f)\|^2 + \|S_N(f)\|^2$ [Pythagoras]

$$\|V_N(f)\|^2 = \|\sum_{|n|\leq N} \langle f, e_n\rangle e_n\|^2$$
$$= \sum_{|n|\leq N} |\hat{f}(n)|^2$$

?

$$V_N(f) = \sum_{|n|\leq N} \langle f, V_n\rangle V_n = \sum_{|n|\leq N} |\langle f, V_n\rangle|^2$$

$$\|f\|^2 = \|f - V_N(f)\|^2 + \sum_{|n|\leq N} |\langle f, V_n\rangle|^2$$

$$\boxed{\|f\|^2 \geq \sum_{|n|\leq N} |\langle f, V_n\rangle|^2} \longrightarrow \{V_n\} \text{ any orthonormal family.}$$

BESSEL INEQUALITY.

Cauchy Schwarz Inequality

$\ell^2(\mathbb{Z})$

$\langle A, B \rangle = \sum_{n \in \mathbb{Z}} a_n \overline{b_n}$

$|\langle A, B \rangle|^2 \leq \|A\|^2 \|B\|^2$

$$\Big| \sum_{n \in \mathbb{Z}} a_n \overline{b_n} \Big|^2 \leq \Big(\sum |a_n|^2 \Big) \Big(\sum |b_n|^2 \Big)$$

$$\mathcal{R}: \quad (f, g) = \frac{1}{2\pi} \int_0^{2\pi} f(x) \overline{g(x)} \, dx$$

↳ V. space: norm + i.p. → C–S works!

C-S:

$$\left| \frac{1}{2\pi} \int_0^{2\pi} f(x) \overline{g(x)} \right|^2 \leq \|f\|^2 \|g\|^2.$$

Can prove: $$\frac{1}{2\pi} \int_0^{2\pi} |f(x) \overline{g}(x)| \, dx \leq \|f\| \cdot \|g\|$$

[Using: $2AB \leq A^2 + B^2$]

Prove this stronger version of C-S.

Can prove: $$\sum |a_n \overline{b_n}| \leq \Big(\sum |a_n|^2 \Big)^{1/2} \Big(\sum |b_n|^2 \Big)^{1/2}$$

Can pick: $a_n = 0 \ \forall |n| \geq N$:

$$\text{C-S:} \quad \underbrace{\sum_{|n|\leq N} |a_n c_n| \leq \Big(\sum_{|n|\leq N} |a_n|^2\Big)^{1/2} \Big(\sum_{|n|\leq N} |c_n|^2\Big)^{1/2}}_{\text{This is "our" C-S ineq.}} \quad \forall c_n \in \mathbb{C}.$$

THEOREMS:

1) f cont. on the circle

$$\sum_{n\in\mathbb{Z}} |\hat{f}(n)| \Rightarrow S_N(f) \to f$$

2) f is twice cont. differ $\Rightarrow S_N(f) \to f$

3) f is integrable on the circle $\Rightarrow \|S_N(f) - f\| \to 0$.

$f: \mathbb{R} \to \mathbb{R}$ 2π-periodic and $f''(x) \ \forall x$ is cont.

$f: [-\pi, \pi] \to \mathbb{R}$ and $f(-\pi) = f(\pi)$. $\left.\begin{matrix} f''_+(-\pi) \\ f''_-(\pi) \end{matrix}\right\}$ one-sided derivatives.

4) $f: \mathbb{R} \to \mathbb{C}$, 2π-periodic

$$f \in C^1(\mathbb{R}) \Rightarrow S_N(f)(\theta) \to f(\theta) \quad \forall \theta \in \mathbb{R}.$$

Can prove using Parseval.

THIS IS NOT IN BOOK!

$n \in \mathbb{Z}$, $n \neq 0$:

$$\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta) e^{-in\theta} d\theta = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\left(\frac{e^{-in\theta}}{-in}\right)' d\theta$$

$$= \frac{1}{2\pi}\Bigg\{\underbrace{f(2\pi)\frac{e^{-in2\pi}}{-in} - f(0)\frac{e^{-in0}}{-in}}_{\substack{=0 \text{ b/c } f(2\pi)=f(0) \\ \text{and } e^{-in2\pi} = e^{-in0} = 1}}\Bigg\} - \int_0^{2\pi} f'(\theta)\left(\frac{e^{-in\theta}}{-in}\right) d\theta$$

$$= \frac{1}{2\pi i n} \underbrace{\int_0^{2\pi} f'(\theta) e^{-in\theta} d\theta.}_{= \widehat{f'}(n)}$$

$$\boxed{in \cdot \hat{f}(n) = \widehat{f'}(n)} \quad \text{or: } \hat{f}(n) = \frac{1}{in}\widehat{f'}(n)$$

$$\sum_{\substack{|n| \leq N \\ n \neq 0}} |\hat{f}(n)| = \sum_{\substack{|n| \leq N \\ n \neq 0}} \frac{|\widehat{f'}(n)|}{|n|} \overset{\text{C-S ineq.}}{\leq} \left(\sum_{\substack{|n| \leq N \\ n \neq 0}} |\widehat{f'}(n)|^2\right)^{1/2} \left(\sum_{\substack{|n| \leq N \\ n \neq 0}} \frac{1}{|n|^2}\right)^{1/2}$$

So: $$\sum_{\substack{|n| \leq N \\ n \neq 0}} |\hat{f}(n)| \leq \left(\sum_{\substack{|n| \leq N \\ n \neq 0}} |\widehat{f'}(n)|^2\right)^{1/2} \cdot B \leq \|f'\| \cdot B$$

<u>Parseval</u>: $\sum_{n \in \mathbb{Z}} |\hat{g}(n)|^2 = \|g\|^2$ Pick now $g = f'$:

$$\sum_{\substack{|n|\leq N \\ n\neq 0}} |\widehat{f'}(n)|^2 \leq \|f'\|^2$$

$$\Rightarrow \sum_{n\in\mathbb{Z}} |\hat{f}(n)| < \infty$$

By Thm 1

$$\Rightarrow S_N(f) \to f.$$

RIEMANN – LEBESGUE LEMMA

Let f integrable on the circle. Then:

$$\lim_{n\to\pm\infty} \hat{f}(n) = 0.$$

Proof: Using Parseval's identity: $\|f\|^2 = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2$

this sum must converge. b/c equal to $\|f\|^2$.

Recall: $\sum a_n < \infty \Rightarrow \lim_{n\to\infty} a_n = 0.$

Can write $S_N = \sum_{n=1}^{N} a_n$

$S_N - S_{N-1} = a_N$ as $N\to\infty$: $S - S = \lim_{n\to\infty} a_n$

$0 = \lim_{n\to\infty} a_n$

$$\|f\|^2 = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2 \Rightarrow \sum_{n=1}^{\infty} |\hat{f}(n)|^2 + \sum_{n=-\infty}^{0} |\hat{f}(n)|^2$$

Both sums exist and are convergent.